Chemical algebra. VII: Improper G-weighted metrics of non-compact groups: Lorentz group in the Minkowski space

Remi Chauvin

Laboratoire de Chimie de Coordination du C.N.R.S., Unité 8241, liée par conventions à l'Université Paul Sabatier et à l'Institut National Polytechnique, 205 Route de Narbonne, 31 077 Toulouse Cedex, France

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The principles and the generalized equation of chemical algebra is extended to a Minkowskian substrate E endowed with its improper non-definite-positive metric, where the non-compact 6-parameter group G of the Lorentz transformations operates. Given a map $\mu_{\mathbf{u},\mathbf{u}}(g) = \mu(g\mathbf{u})m(\mathbf{g})$ on G, a "line element" ds^2 is formulated at each point marked by a vector $\mathbf{u}$. Assuming "$\mu = 1$" and "$m(g) \neq 0 \Rightarrow g$ is a pure Lorentz transformation (without a spatial rotation)", the isotropic hypothesis (m depends on a single parameter out of three in G) is first studied. In general, ds^2 does not define a Riemannian manifold unless one additional condition on m is imposed. Several relationships are established which are useful for the calculation of the metric tensor and the curvature tensor.

1. Introduction

The more general starting material of the propositions of chemical algebra established hitherto is a "substrate", i.e. a metric space endowed with a group of isometric transformations. Our attempt to apply chemical algebra to the classical Euclidean space has led to a remarkably concise formulation of a ds^2 as the exact second logarithmic differential of a scalar "wave function" associated to a Schrödinger-type equation [1]. In the sequel, the study of the space-time of relativity is launched. This space-time is not represented by a metric space in the classical sense: the improper "distance" between two distinct events is not necessarily positive nor different from zero. In other words, the underlying bilinear form of $\mathbb{R}^4$ is not definite-positive. The first task is to find out how to formulate the definition of a pairing product on such a space. Then, by analogy with our previous treatment [1], we shall consider some group acting isometrically in $\mathbb{R}^4$, and transitively in one part of the projective space $\mathbf{P}(\mathbb{R}^4)$ (contrary to translations which are not linear, the consideration of the projective space is needed for *linear* groups cannot act transitively in $\mathbb{R}^4$: $\{\mathbf{0}\}$ is an orbit).

It is already obvious that spatial rotations are not sufficient. In addition, this group must correspond to "physical motions". From a relativistic viewpoint, the displacement of the set trihedron + clock can merely be performed with a finite velocity $\mathbf{v}$ with respect to its initial situation ($|\mathbf{v}| < c$), and brings about a slowing down of the clock. This slowing down is correlated to $\mathbf{v}$ and the admissible transformations constitute the Lorentz group. The present report deals with the formulation of the chemical algebra on the Minkowski space, whose metric is defined by the pseudonorm $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ and with the Lorentz group as an isometry group which acts transitively inside the projective light cone ($c^2t^2 - |\mathbf{r}|^2 > 0$). Although the process is only mathematical in nature, the terminology of physics ("space", "time", "velocity", etc.) is used for the sake of brevity. In order to clarify the process, the layout of this article is summarized below:

2. Generalized equation for improper metric substrates

2.1. EXPRESSION OF THE PAIRING PRODUCT $K_p(\mathbf{u}, \mathbf{u} + d\mathbf{u})$

If (E, d) is a metric space, then,

$$K_p^p(\mathbf{u}, \mathbf{v}) = \frac{\int_G \mu_{\mathbf{u},\mathbf{v}}(g) \exp\left[-\frac{p}{2} d^2(g\mathbf{u}, \mathbf{u})\right] dg \cdot \int_G \mu_{\mathbf{u},\mathbf{v}}(g) \exp\left[-\frac{p}{2} d^2(g\mathbf{v}, \mathbf{v})\right] dg}{\int_G \mu_{\mathbf{u},\mathbf{v}}(g) \exp\left[-\frac{p}{2} d^2(g\mathbf{u}, \mathbf{v})\right] dg \cdot \int_G \mu_{\mathbf{u},\mathbf{v}}(g) \exp\left[-\frac{p}{2} d^2(g\mathbf{v}, \mathbf{u})\right] dg}.$$

If E is a real vector space and if d is the Euclidean distance derived from a definite-positive symmetric bilinear form, this definition also reads

$$K_p^p(\mathbf{u},\mathbf{v}) = \frac{\int_G \mu_{\mathbf{u},\mathbf{v}}(g)\exp[p(g\mathbf{u}|\mathbf{u})]\,dg \cdot \int_G \mu_{\mathbf{u},\mathbf{v}}(g)\exp[p(g\mathbf{v}|\mathbf{v})]\,dg}{\int_G \mu_{\mathbf{u},\mathbf{v}}(g)\exp[p(g\mathbf{u}|\mathbf{v})]\,dg \cdot \int_G \mu_{\mathbf{u},\mathbf{v}}(g)\exp[p(g\mathbf{v}|\mathbf{u})]\,dg}\,.$$

This definition can be naturally retained for *any symmetric bilinear form*. The corresponding map $d(\mathbf{u},\mathbf{v}) = \sqrt{(\mathbf{u}-\mathbf{v}|\mathbf{u}-\mathbf{v})} \in \mathbb{C}$ is no longer a distance, but is relevant in the definition of pseudo-Euclidean spaces (e.g. the Minkowski space).

Remark

An alternative generalization of the definition used hitherto for proper Euclidean spaces could call for the modulus of the exponential terms occurring in the integrands instead of the "nacked" exponentials, that is for $\exp[p\,\mathrm{Re}(g\mathbf{u}|\mathbf{u})]$ instead of $\exp[p(g\mathbf{u}|\mathbf{u})]$. Indeed, the former expression afforded a suitable definition equation on Hermitean complex vector spaces [2]. Nevertheless, the vector space substrates to be considered below are real, and the more direct generalization is considered *a priori*.

2.2. GENERAL EXPRESSION OF ds^2.

From the pairing product $K_p(\mathbf{u},\mathbf{v})$, a general equation ($\mathbb{E}$) for the definition of a distance $D_p(\mathbf{u},\mathbf{v})$ has been devised

$$\Phi_{\mathbf{u},\mathbf{v}}(D_p(\mathbf{u},\mathbf{v})) = K_p(\mathbf{u},\mathbf{v})\,, \qquad (\mathbb{E})$$

where $\Phi_{\mathbf{u},\mathbf{v}}(x)$ is one numerical map which has been formulated on the basis of a set of consistency requirements [3].

If G is a finite group, eq. ($\mathbb{E}$) can be applied to any pair of vectors $\mathbf{u}$ and $\mathbf{v}$ of a connected part in a vector space E (called "substrate") leading to a G-weighted distance extension $D_p(\mathbf{u},\mathbf{v})$ on E (if the weighting map $\mu_{\mathbf{u},\mathbf{v}}(g)$ is constant, D_p is a completely G-invariant distance extension on E/G) [4]. Alternatively, ($\mathbb{E}$) can be first applied to infinitely close vectors $\mathbf{w}$ and $\mathbf{w}+d\mathbf{w}$ leading to a G-weighted metric ds^2 at each point $\mathbf{w}$: the minimum integral of ds along all possible pathways joining $\mathbf{u}$ and $\mathbf{v}$ (if it exists) leads to another distance $L_p(\mathbf{u},\mathbf{v})$. When G is a non-compact group such that $D_p(\mathbf{u},\mathbf{v}) = \infty$ if $\mathbf{u} \neq \mathbf{v}$, the value $D_p(\mathbf{u},\mathbf{u}) = 0$ still holds. Thus, when $\mathbf{v} = \mathbf{u} + d\mathbf{u}$, the crude equation has been extended in order to define relevant values of $ds^2 = D_p^2(\mathbf{u},\mathbf{u}+d\mathbf{u})$ "somewhere between zero and infinity" [5]. If $\mathbf{v} = \mathbf{u}+d\mathbf{u}$, the expression of the pairing product can be simplified very generally. A somewhat tedious calculation leads to

$$K_p(\mathbf{u},\mathbf{u}+d\mathbf{u}) \approx 1 + \frac{p}{I}\{d^2K + \frac{p}{I}(I\cdot d^2L - dJdJ')\}\,,$$

where

$$I = \int_G \mu_{\mathbf{u},\mathbf{u}}(g) \cdot \exp\left[-\frac{p}{2}\|g\mathbf{u} - \mathbf{u}\|^2\right] dg\,,$$

$$d^2K = \int_G \mu_{\mathbf{u},\mathbf{u}}(g) \cdot (gd\mathbf{u}|d\mathbf{u}) \cdot \exp\left[-\frac{p}{2}\|g\mathbf{u} - \mathbf{u}\|^2\right] dg\,,$$

$$d^2L = \int_G \mu_{\mathbf{u},\mathbf{u}}(g) \cdot (g\mathbf{u}|d\mathbf{u}) \cdot (gd\mathbf{u}|\mathbf{u}) \cdot \exp\left[-\frac{p}{2}\|g\mathbf{u} - \mathbf{u}\|^2\right] dg\,,$$

$$dJ = \int_G \mu_{\mathbf{u},\mathbf{u}}(g) \cdot (g\mathbf{u}|d\mathbf{u}) \cdot \exp\left[-\frac{p}{2}\|g\mathbf{u} - \mathbf{u}\|^2\right] dg\,,$$

$$dJ' = \int_G \mu_{\mathbf{u},\mathbf{u}}(g) \cdot (gd\mathbf{u}|\mathbf{u}) \cdot \exp\left[-\frac{p}{2}\|g\mathbf{u} - \mathbf{u}\|^2\right] dg$$

(the notations "d" and "d^2" denote formal differential forms: in rigorous notations used in ref. [5], $K_p(\mathbf{u}, \mathbf{u} + d\mathbf{u}) = 1 + \frac{1}{2}d^2K_0^{\mathbf{u}}$).

It is already evident that $dJ = dJ'$ as soon as $\mu_{\mathbf{u},\mathbf{u}}(g) = \mu_{\mathbf{u},\mathbf{u}}(g^{-1})$ over G.

3. *G*-weighted metrics of the Lorentz group on a Minkowskian substrate

3.1. ACTION OF THE LORENTZ GROUP

Whereas the group of translations in $\mathbb{R}^3$ is a 3-parameter group, the restricted Lorentz group is a 6-parameter group, but a Lorentz transformation which does not involve a spatial rotation is characterized by three numbers v (or $\beta = v/c$), θ and ϕ. These numbers define the modulus and the direction of the relative velocity $\mathbf{v}$ of two trihedron–clock systems: $0 \leqslant \beta \leqslant 1$, $0 < \theta < \pi$, $0 \leqslant \phi < 2\pi$. In a more abstract way, these transformations are "hyperbolic rotations" in the Minkowski space with one plane containing the time axis. It is to be verified that if (x, y, z, ct) is a 4-vector and if $(x', y', z', ct') = g(x, y, z, ct)$, then,

$$\begin{aligned} x' &= [1 + (\gamma - 1)\sin^2\theta\cos^2\phi]x + (\gamma - 1)\sin^2\theta\sin\phi\cos\phi\, y \\ &\quad + (\gamma - 1)\sin\theta\cos\theta\cos\phi\, z - \gamma\sin\theta\cos\phi\beta\, ct\,, \end{aligned}$$

$$\begin{aligned} y' &= (\gamma - 1)\sin^2\theta\sin\phi\cos\phi\, x + [1 + (\gamma - 1)\sin^2\theta\sin^2\phi]\, y \\ &\quad + (\gamma - 1)\sin\theta\cos\theta\sin\phi\, z - \gamma\sin\theta\sin\phi\,\beta\, ct\,, \end{aligned}$$

$$\begin{aligned} z' &= (\gamma - 1)\sin\theta\cos\theta\cos\phi\, x + (\gamma - 1)\sin\theta\cos\theta\sin\phi\, y \\ &\quad + [1 + (\gamma - 1)\cos^2\theta]z - \gamma\cos\theta\,\beta\, ct; \end{aligned}$$

$$ct' = -\gamma \sin\theta \cos\phi\, \beta x - \gamma \sin\theta \sin\phi\, \beta y - \gamma \cos\theta\, \beta z + \gamma ct,$$

where: $\beta = \mathrm{v/c}$ and $\gamma = [1 - \beta^2]^{-1/2}$

On the other hand, a short calculation leads to

$$\|g\mathbf{u} - \mathbf{u}\|^2 = 2(1-\gamma)[(ct)^2 - (\sin\theta\cos\phi\, x + \sin\theta\sin\phi\, y + \cos\theta\, z)^2],$$

where g is a Lorentz transformation (characterized by a vector $\mathbf{v}$), $\mathbf{u} = (x, y, z, ct)$ is a 4-vector, and $\|\cdot\|$ denotes the pseudonorm of the Minkowski space.

Remark 1

It cannot be overemphasized that in the above formulae, g denotes a pure Lorentz transformation without a component of spatial rotation. Any element of the Lorentz group is written gk, where k is a pure spatial rotation. Nevertheless, this general writing will be inessential in the sequel, for it will be supposed that $\mu_{\mathbf{u},\mathbf{u}}(gk) \neq 0$ only if $k = \mathrm{e}$.

Remark 2

The above squared "distance" between $g\mathbf{u}$ and $\mathbf{u}$ can be complex or real, positive or negative. It is *real* (like γ) as soon as $v \leqslant c$.

Remark 3

Contrary to the case of translations in the Euclidean space where $\|g\mathbf{u} - \mathbf{u}\|^2$ did not depend on $\mathbf{u}$, $\|g\mathbf{u} - \mathbf{u}\|^2$ now depends on both g and $\mathbf{u}$.

Remark 4

When $v << c$, i.e. when $\beta \to 0$, then,

$$\|g\mathbf{u} - \mathbf{u}\|^2 \approx 2(1-\gamma)(ct)^2 \approx 2\left(-\frac{1}{2}\frac{v^2}{c^2}\right)c^2t^2 \approx -(vt)^2,$$

i.e. $\|g\mathbf{u} - \mathbf{u}\|^2 \to -(vt)^2$: we find again the expression of $\|g\mathbf{u} - \mathbf{u}\|^2$ (except the sign minus, as expected) for the Euclidean space where Lorentz transformations are replaced by translations g characterized a vector $\mathbf{v}t$.

Remark 5

The Lorentz group (Lie group of dimension 6) does not act transitively in the projective space $\mathbf{P}(\mathbb{R}^4)$ (there are two orbits of lines), but it acts transitively in the orbit corresponding to the interior of the light cone ($c^2t^2 - |\mathbf{r}|^2 > 0$). In the sequel, $\mathbf{u}$ denotes the coordinates of any point inside this region.

3.2. INTEGRALS ON THE LORENTZ GROUP

On the very outset, the measure dg serving to define ds^2 is identified with a volume element in the velocity space time the Haar measure of the group of spatial rotations $dk = 1/8\pi^2 \sin\beta' \, d\alpha' d\beta' d\gamma'$ (where α', β', γ' are Euler's angles). The relativistic velocity space is endowed with the metric (in spherical coordinates):

$$dl_v^2 = \gamma^4 dv^2 + \gamma^2 v^2 d\theta^2 + \gamma^2 v^2 \sin^2\theta \, d\phi^2 .$$

The volume element of this space is proportional to the square root of the modulus of the (diagonal) tensor matrix time $dv d\theta d\phi$. Thus, taking the proportionality factor equal to $3/4\pi c^3$ (in order to normalize the volume of the "sphere" of physical velocities: $v \leqslant c$),

$$dg = \frac{3}{4\pi c^3}\sqrt{\gamma^4 \cdot \gamma^2 v^2 \cdot \gamma^2 v^2 \sin^2\theta} \, dv d\theta d\phi \, dk = \frac{3}{4\pi}\gamma^4 \beta^2 \sin\theta \, d\beta d\theta d\phi \, dk .$$

From a set-theoretic standpoint, the Lorentz group is regarded as a part of $\mathbb{R}^6$ (6-parameter group). The integral symbol $\int_G \ldots dg$ sweeps all the possible directions of the velocity $\mathbf{v}$ ($0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi < 2\pi$) and all the possible Euler's angles ($0 \leqslant \alpha' \leqslant 2\pi$, $0 \leqslant \beta' \leqslant \pi$, $0 \leqslant \gamma' \leqslant 2\pi$). Regarding the modulus of $\mathbf{v}$, G might gather all the possible values from 0 to $+\infty$ instead of being restricted to physical velocities, i.e. to $0 \leqslant v \leqslant c$ (the composition of two operations $\mathbf{v}$ and $\mathbf{v}'$ such that $v \leqslant c$, $v' \leqslant c$ is an operation $\mathbf{v}''$ such that $v'' \leqslant c$). Whereas the domain $0 \leqslant \beta < \infty$ makes G operate in the complexified vector space of E (a real event can be transformed into a non-real one), the domain $0 \leqslant \beta \leqslant 1$ ensures that G operates inside the real vector space E (and inside the light cone). The latter restriction reflects the requirement that if $\mathbf{u}$ is an event (with *real* space and time coordinates), $g\mathbf{u}$ is an event as well, for γ is then real.

Finally, it is to be reminded that G is not compact:

$$\int_G dg = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi} \sin\beta' \, d\alpha' \, d\beta' \, d\gamma' \cdot \frac{3}{4\pi c^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{c} \gamma^4 v^2 \sin\theta$$
$$\times \, dv \, d\theta \, d\phi = 1.3 \int_0^1 \frac{\beta^2}{(1-\beta^2)^2} d\beta = +\infty .$$

3.3. FORM OF THE WEIGHTING MAP $\mu_{\mathbf{u},\mathbf{u}}$

At the outset, it has been supposed that, apart from the algebraic datum of an action of G onto E, no quantitative link is presupposed between G and E. Therefore, $\mu_{\mathbf{u},\mathbf{u}}$ should have the form $\mu_{\mathbf{u},\mathbf{u}}(g) = m(g)\mu(g\mathbf{u})$, where m and μ are independent weighting functions of G and E, respectively. In our search for G-weighted metrics on the Euclidean space where G is the group of translations, it has been assumed that $m = 1$. Here, although it might be interesting, the same assumption does not seem to allow for straightforward simplifications [6]. In the sequel, we are first concerned with the symmetric assumption $\mu = 1$. The latter hypothesis has been the starting motivation for generalizing the definition equation of completely G-

invariant distances to the definition equation of G-weighted distances, the two definitions meeting when $\mu_{\mathbf{u},\mathbf{v}} = 1$. The map $\mu_{\mathbf{u},\mathbf{u}} = m$ can be regarded as the characteristic function of a fuzzy subset of the Lorentz group [4,7]. From now onwards and until section 5, it is assumed that:

$$\mu_{\mathbf{u},\mathbf{u}}(g) \neq 0 \text{ only if } g \text{ is a Lorentz transformation,}$$
$$\text{i.e. if no spatial rotation is involved.}$$

In other words, the integral domains can be restricted to only three parameters out of six: v, θ and ϕ. This hypothesis allows us for keeping one strong analogy with the case of the 3-parameter group of translations in the 3D-Euclidean space (rotations were not considered either). This analogy might be interesting owing to the finding that the metric ds^2 was completely determined by a solution of the Schrödinger operator ("wave function").

3.4. FORMULATION OF G-WEIGHTED METRICS ON A MINKOWSKIAN SUBSTRATE

When γ is real (i.e. when $v<c$), then $1-\gamma<0$. Since $\|g\mathbf{u}-\mathbf{u}\|^2 = 2(1-\gamma) \times[(ct)^2 - (\sin\theta\cos\phi\, x + \sin\theta\sin\phi\, y + \cos\theta\, z)^2]$, the real terms $-(p/2)\|g\mathbf{u}-\mathbf{u}\|^2$ in the exponentials are negative if and only if $p<0$ for $r<ct$ and $p>0$ for $r>ct$: this is a necessary condition to get convergent integrals with standard maps $\mu_{\mathbf{u},\mathbf{u}}$ (e.g. $\mu_{\mathbf{u},\mathbf{u}} = 1$). Possible interpretations of p will be discussed later, but the condition $p<0$ seems to be required inside the light cone (along the time axis), while the opposite condition ($p>0$, which has been hitherto required for standard distances and Euclidean metrics) would have been required outside the light cone.

Given a time coordinate $x^0 = ct$ and three orthogonal spatial coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$, a Lorentz transformation g (without a spatial rotation) in this coordinate system is given by a 4,4-matrix $M(g) = (a^i_j(g))_{\substack{0\leqslant i\leqslant 3\\ 0\leqslant j\leqslant 3}}$: the coefficients a^i_j are given by the above relationships $x'^i = \sum_{j=1}^n a^i_j(g)x^j$ (the a^i_j's depend on the parameters of g, namely: v, θ and ϕ). g acts as a coordinate transformation, and $M(g)$ is a symmetric matrix, i.e.: $a^i_j = a^j_i$. Since $a^i_j(g) = \partial x'^i/\partial x^j$, $\{x^i\}$ and $\{x'^i\}$ denote the coordinates of one *contravariant* 4-vector in two "rectangular coordinate systems". Assuming the definitions $x_0 = x^0$, $x_1 = -x^1$, $x_2 = -x^2$, $x_3 = -x^3$ and $x'_0 = x'^0$, $x'_1 = -x'^1$, $x'_2 = -x'^2$, $x'_3 = -x'^3$, $\{x_i\}$ and $\{x'_i\}$ denote the coordinates of one *covariant* vector. Indeed, the covariant set of equations $\{x'^i = a^i_j(g)x^j\}$, namely,

$$x'^0 = a^0_0(g)x^0 + a^0_\alpha(g)x^\alpha,$$

$$x'^\beta = a^\beta_0(g)x^0 + a^\beta_\alpha(g)x^\alpha$$

can be written as

$$x'_0 = a_0^0(g)x_0 - a_0^\alpha(g)x_\alpha\,,$$

$$-x'_\beta = a_\beta^0(g)x_0 - a_\beta^\alpha(g)x_\alpha\,.$$

By changing the group parameter β for $-\beta$ to get $a_j^i(g^{-1})$ from $a_j^i(g)$, it is verified that $a_0^0(g) = a_0^0(g^{-1})$, $-a_\alpha^0(g) = a_\alpha^0(g^{-1})$, $a_0^\beta(g) = -a_0^\beta(g^{-1})$, $a_\alpha^\beta(g) = a_\alpha^\beta(g^{-1})$. Thus, $x'_i = a_i^j(g^{-1})x_j$. Since $a_i^j(g^{-1}) = \partial x^j/\partial x'^i$, $\{x_i\}$ transforms as a covariant vector under the Lorentz transformation g of the reference contravariant coordinates.

Thus, we calculate the "inner products":

$$(g d\mathbf{u}|d\mathbf{u}) = dx'^i dx_i = dx'^0 dx^0 - dx'^1 dx^1 - dx'^2 dx^2 - dx'^3 dx^3 = a_{ij}(g) dx^i dx^j\,,$$

$$(g\mathbf{u}|d\mathbf{u}) = x'_i dx^i = x'^0 dx^0 - x'^1 dx^1 - x'^2 dx^2 - x'^3 dx^3 = a_{ij}(g) x^j dx^i\,,$$

$$(g d\mathbf{u}|\mathbf{u}) = dx'_i x^i = dx'^0 x^0 - dx'^1 x^1 - dx'^2 x^2 - dx'^3 x^3 = a_{ij}(g) x^i dx^j\,,$$

where $a_{0j} = a_j^0$ and $a_{\alpha j} = -a_j^\alpha$, $\alpha = 1, 2, 3$. Thus,

$$(a_{ij}) = \begin{pmatrix} -[1+(\gamma-1)\sin^2\theta\cos^2\phi] & -[(\gamma-1)\sin^2\theta\sin\phi\cos\phi] & -[(\gamma-1)\sin\theta\cos\theta\cos\phi] & \gamma\sin\theta\cos\phi\beta \\ -[(\gamma-1)\sin^2\theta\sin\phi\cos\phi] & -[1+(\gamma-1)\sin^2\theta\sin^2\phi] & -[(\gamma-1)\sin\theta\cos\theta\sin\phi] & \gamma\sin\theta\sin\phi\beta \\ -[(\gamma-1)\sin\theta\cos\theta\cos\phi] & -[(\gamma-1)\sin\theta\cos\theta\sin\phi] & -[1+(\gamma-1)\cos^2\theta] & \gamma\cos\theta\beta \\ -\gamma\sin\theta\cos\phi\beta & -\gamma\sin\theta\sin\phi\beta & -\gamma\cos\theta\beta & \gamma \end{pmatrix}.$$

Indeed, the starting coordinates x^i (but not necessarily the x'^i's if $\beta \geqslant 1$) are real. If we put

$$I_{ij} = \int_G \mu_{\mathbf{u},\mathbf{u}}(g) a_{ij}(g) \exp\left[-\frac{p}{2}\|g\mathbf{u}-\mathbf{u}\|^2\right] dg\,,$$

and

$$L_{iklj} = \int_G \mu_{\mathbf{u},\mathbf{u}}(g) a_{ik}(g) a_{lj}(g) \exp\left[-\frac{p}{2}\|g\mathbf{u}-\mathbf{u}\|^2\right] dg\,,$$

then,

$$d^2K = I_{ij} dx^i dx^j\,,$$

$$dJ = I_{ij} x^j dx^i\,, \quad dJ' = I_{ij} x^i dx^j \quad \text{and thus} \quad dJdJ' = I_{ik} I_{lj} x^k x^l dx^i dx^j\,,$$

$$d^2L = L_{iklj} x^k x^l dx^i dx^j\,.$$

It follows that

$$\begin{aligned} K_p(\mathbf{u}, \mathbf{u}+d\mathbf{u}) &\approx 1 + \frac{p}{I}\left\{I_{ij} dx^i dx^j + \frac{p}{I}(I \cdot L_{iklj} x^k x^l dx^i dx^j - I_{ik} I_{lj} x^k x^l dx^i dx^j)\right\} \\ &\approx 1 + \frac{p}{I}\left\{I_{ij} + \frac{p}{I}(I \cdot L_{iklj} - I_{ik} I_{lj}) x^k x^l\right\} dx^i dx^j\,. \end{aligned}$$

Contrary to the group of translations in $\mathbb{R}^n$, the Lorentz group G acts linearly in $\mathbb{R}^4$. The formula

$$\Phi_{\mathbf{u},\mathbf{u}+d\mathbf{u}}(ds\gamma) \approx 1 + pB^2(\mathbf{u}, d\mathbf{u})ds^2$$

can be used to formulate the basic equation ($\mathbb{E}$) serving to define a G-weighted metric ds^2 from $K_p(\mathbf{u}, \mathbf{u} + d\mathbf{u})$ [4]. For a real bilinear form, B^2 is given by

$$B^2(\mathbf{u}, d\mathbf{u}) = \frac{1}{\left[\int_G \mu_{\mathbf{u},\mathbf{u}}(g)\, dg\right]^4} \frac{\|d\mathbf{U}\|^2}{\|d\mathbf{u}\|^2},$$

where

$$d\mathbf{U} = \int_G \mu^2_{\mathbf{u},\mathbf{u}}(g)(gd\mathbf{u})\, dg.$$

In conclusion,

$$ds^2 = \frac{1}{I^2 \cdot B^2}\{I \cdot I_{ij} + p(I \cdot L_{iklj} - I_{ik} \cdot I_{lj})x^k x^l\}dx^i dx^j,$$

where B^2 stands for $B^2(\mathbf{u}, d\mathbf{u})$.

It is noteworthy, that if $\mu = 1$, then $B^2 = 0$ and no definition of ds^2 is obtained from ($\mathbb{E}$) (instead of ds^2 we might get a formal definition of ds^4).

4. Isotropic ds^2 of the Lorentz group

The title term means that if g is a Lorentz transformation defined by v, θ, ϕ, the membership degree of g to the considered fuzzy subgroup is independent of the direction of the velocity $\mathbf{v}$, i.e. of both θ and ϕ. Thus, we may write $m(g) = m(v)$.

4.1. K_p-DERIVED INTEGRALS: I, I_{ij} and L_{iklj}

- Calculation of $I = \int_G m(v) \exp\left[-\frac{p}{2}\|g\mathbf{u} - \mathbf{u}\|^2\right] dg$

$$I = \frac{3}{4\pi c^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{c} m(v) \exp[p(\gamma - 1)\{(ct)^2 - (\sin\theta\cos\phi\, x + \sin\theta\sin\phi\, y + \cos\theta\, z)^2\}]\gamma^4 v^2 \sin\theta\, dv\, d\theta\, d\phi.$$

Let $k(\theta, \phi) = (ct)^2 - (\sin\theta\cos\phi\, x + \sin\theta\sin\phi\, y + \cos\theta\, z)^2 = (ct)^2 - (\mathbf{r}\cdot\mathbf{v}/v)^2 = (ct)^2 - r^2\cos^2(\mathbf{n}_\mathbf{v}, \mathbf{r})$ where $\mathbf{n}_\mathbf{v}$ is the unit vector normal to the sphere or radius 1 and parallel to $\mathbf{v}$ (k does not vary with the modulus v). If we put $I = 3\int_0^1 m(c\beta)I_a(\beta)\beta^2\gamma^4\, d\beta$, the integral I_a on (θ, ϕ) reads

$$I_a = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \exp[p(\gamma-1)k(\theta,\phi)]\sin\theta\, d\theta\, d\phi$$
$$= \frac{1}{4\pi}\exp[p(\gamma-1)(ct)^2]\iint_{S^\circ} \exp[-p(\gamma-1)r^2\cos^2(\mathbf{n}_\mathbf{v},\mathbf{r})]\, dS\,.$$

The integral over the whole sphere S° does not depend on the direction of $\mathbf{r}$ (I and I_a are "scalar tensors" for linear Lorentz transformations). It is calculated for $\mathbf{r}$ along the z-axis ($x = y = 0$ and $z = r$), for then $(\mathbf{n}_\mathbf{v},\mathbf{r}) = \theta$:

$$I_a = \frac{1}{2}\exp[p(\gamma-1)(ct)^2]\int_0^{\pi} \exp[-p(\gamma-1)r^2\cos^2\theta]\sin\theta\, d\theta$$
$$= \exp[p(\gamma-1)(ct)^2]\int_0^{1} \exp[-p(\gamma-1)r^2u^2]\, du\,.$$

Therefore,

$$I = 3\exp[p(\gamma-1)(ct)^2]\int_0^1\int_0^1 m(c\beta)\exp[-p(\gamma-1)r^2u^2]\beta^2\gamma^4\, d\beta\, du\,.$$

- Calculation of $I_{ij} = \dfrac{3}{4\pi}\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{1} m(c\beta)a_{ij}(g)\exp[pk(\theta,\phi)(\gamma-1)]\gamma^4\beta^2\sin\theta \times d\beta\, d\theta\, d\phi$, and

$$L_{iklj} = \frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{1} m(c\beta)a_{ik}(g)a_{lj}(g)\exp[pk(\theta,\phi)(\gamma-1)]\gamma^4\beta^2\sin\theta\, d\beta\, d\theta\, d\phi$$

Contrary to I, these integrals are not scalar tensors: whereas $k(\theta,\phi) = (ct)^2 - r^2\cos^2(\mathbf{n}_\mathbf{v},\mathbf{r})$ depends on both (θ,ϕ) and $\mathbf{u}$, the coefficients $a_{ij}(g)$ or $a_{ik}(g)a_{lj}(g)$ do not depend on $\mathbf{u}$ but depend explicitly on (θ,ϕ). Changing the integrand variable $\mathbf{n}_\mathbf{v}$ for the unit vector along the $\mathbf{r}$-axis does not lead to an obvious simplification.

4.2. THE ϕ-DERIVED INTEGRALS

- Calculation of $B^2(\mathbf{u}, d\mathbf{u}) = \dfrac{1}{[\int_G \mu_{\mathbf{u},\mathbf{u}}(g)\, dg]^4}\dfrac{\|d\mathbf{U}\|^2}{\|d\mathbf{u}\|^2}$,

The vector $d\mathbf{U} = \int_G m^2(v)(gd\mathbf{u})\, dg$ can be refered to as its components in contravariant notations:

$$d\mathbf{U}^i = \int_G m^2(v)a^i_j(g)dx^j)\, dg = \int_G m^2(v)a^i_j(g)\, dg\cdot dx^j\,.$$

It is easily verified that all non-diagonal terms $\int_G m^2(v)a^i_j(g)\, dg$ (with $i \neq j$) vanish. Consequently, $dU^i = A_i dx^i$ (no summation on i) with $A_i = \int_G m^2(v)a^i_i(g)\, dg$ (no contraction on i). It is easily verified that

$$A_\alpha = -\frac{1}{c^3}\int_0^c (\gamma+2)m^2(v)\gamma^4v^2\, dv \quad \text{for} \quad \alpha = 1,2,3\,.$$

And

$$A_0 = \frac{3}{c^3}\int_0^c \gamma m^2(v)\gamma^4 v^2\, dv\,.$$

In conclusion,

$$\|d\mathbf{U}\|^2 = A_0^2(dx^0)^2 - A_\alpha^2\{(dx^1)^2 + (dx^2)^2 + (dx^3)^2\}$$

and

$$B^2(\mathbf{u}, d\mathbf{u}) = \frac{A_\alpha^2 + (A_0^2 - A_\alpha^2)(dx^0/\|d\mathbf{u}\|)^2}{\left[\int_G m(v)\, dg\right]^4}\,.$$

4.3. RIEMANNIAN CONDITION

In order that $B^2(\mathbf{u}, d\mathbf{u})$ do not depend on $d\mathbf{u}$, it is necessary that $A_0^2 = A_\alpha^2$, i.e.

$$\int_0^1 (\gamma - 1)m^2(c\beta)\gamma^4\beta^2\, d\beta = 0 \quad \text{or} \quad \int_0^1 (2\gamma + 1)m^2(c\beta)\gamma^4\beta^2\, d\beta = 0\,.$$

Then,

$$m(c\beta) = \frac{1}{v\gamma^2}\left[\frac{f(\beta)}{\gamma - 1}\right]^{1/2} \quad \text{or} \quad \frac{1}{v\gamma^2}\left[\frac{f(\beta)}{2\gamma + 1}\right]^{1/2},$$

where $f(\beta)$ can be any function satisfying $\int_0^1 f(\beta)\, d\beta = 0$ $(\beta = v/c)$. If f is bound to be real-valued, it assumes both positive and negative values: thus $f^{1/2}(\beta)$ and hence $m(c\beta)$ assume both real and imaginary values.

It should be remarked that the first case exhibits a singularity at $\gamma = 1$, i.e. at $\beta = 0$. On the contrary, the pole $\gamma = -1/2$ of the second expression is not attained in the integration domain to be considered $(0 \leqslant \beta \leqslant 1)$.

From the datum of $m(c\beta)$, the other integrals serving to define ds^2 have to be calculated, to wit: $\int_G \mu_{\mathbf{u},\mathbf{u}}(g)\, dg$, $A_\alpha = A_0$, L_{ij} and L_{iklj}. These are, perhaps, neither all real nor all pure imaginary, and the equation would define a non-real metric ds^2, even if $\mathbf{u}$ and $\mathbf{u} + d\mathbf{u}$ mark "real events".

The condition $\int_0^1 f(\beta)\, d\beta = 0$ serving to define a Riemannian G-weighted metric from the map

$$m(c\beta) = \frac{1}{v\gamma^2}\left[\frac{f(\beta)}{\gamma - 1}\right]^{1/2} \quad \text{or} \quad \frac{1}{v\gamma^2}\left[\frac{f(\beta)}{2\gamma + 1}\right]^{1/2}$$

is equivalent to $f(\beta) = F(\beta) - \int_0^1 F(\beta)\, d\beta$, where F is now any integrable map on [0,1].

For instance, if $F(\beta) = \beta$, then $f(\beta) = \beta - \frac{1}{2}$. If $F(\beta) = \gamma$, then $f(\beta) = \gamma - \pi/2$.

Assuming that the corresponding integrals $\int_G m(g)\,dg$, $A_\alpha = A_0$, L_{ij} and L_{iklj} are convergent, the resulting coefficients g_{ij} of ds^2 are certainly not all real. Separating the real and the imaginary parts of ds^2, the equation furnishes two Riemanian metrics, namely $\mathrm{Re}(ds^2)$ and $\mathrm{Im}(ds^2)$, on the same Minkowskian substrate.

In fact, many simple functions $m(c\beta)$ can be found fullfilling the condition $(A_0)^2 = (A_\alpha)^2$, and it seems arbitrary to select a "natural" one. Nevertheless, the general process of calculation can be resumed as follows.

4.4. USE OF A LOCAL COORDINATE SYSTEM FOR ISOTROPIC RIEMANNIAN G-WEIGHTED METRICS

One decides to calculate ds^2 at one point $\mathbf{u}$ on the z-axis: $x = y = 0$, $z = r > 0$. Since the angle ϕ does not appear in the exponential term $(\|g\mathbf{u} - \mathbf{u}\|^2 = 2(1-\gamma)[(ct)^2 - r^2\cos^2\theta])$, the integrals defining ds^2 are much simplified. This study does not entail a loss of generality: indeed, the metric is here supposed to be isotropic, and this choice boils down to selecting a local spatial coordinate system (x, y, z) with the z axis passing through the point M apart from the origin by a distance r (the x, y axes span a plane which is parallel to the tangent plane to the sphere of radius r at M). Thus,

$$I^2B^2ds^2 = \{I \cdot I_{ij} + p(I \cdot L_{i33j} - I_{i3}I_{3j})z^2 + p(I \cdot L_{i00j} - I_{i0}I_{0j})(x^0)^2 + p(I \cdot L_{i30j} - I_{i3}I_{0j} + I \cdot L_{i03j} - I_{i0}I_{3j})zx^0\}dx^i dx^j .$$

It is to be verified that

$I_{ij} \neq 0 \quad \Leftrightarrow \quad i = j,$

$L_{i33j} \neq 0 \quad \Leftrightarrow \quad i = j,$

$L_{i00j} \neq 0 \quad \Leftrightarrow \quad i = j,$

$L_{i30j} \neq 0 \quad \Leftrightarrow \quad (i = 3 \text{ and } j = 0) \text{ or } (i = 0 \text{ and } j = 3)$. Thus, $L_{i30i} = 0$.

$L_{i03j} \neq 0 \quad \Leftrightarrow \quad (i = 3 \text{ and } j = 0) \text{ or } (i = 0 \text{ and } j = 3)$. Thus, $L_{i03i} = 0$.

Consequently,

$$ds^2 = g_{00}c^2dt^2 + g_{11}dx^2 + g_{22}dy^2 + g_{33}dz^2 + 2g_{03}cdtdz ,$$

where

$$I^2B^2g_{00} = I \cdot I_{00} + pI \cdot L_{0330}z^2 + p(I \cdot L_{0000} - I_{00}^2)(ct)^2 ,$$

$$I^2B^2g_{33} = I \cdot I_{33} + p(I \cdot L_{3333} - I_{33}^2)z^2 + pI \cdot L_{3003}(ct)^2 ,$$

$$I^2B^2g_{11} = I \cdot I_{11} + pI \cdot L_{1331}z^2 + pI \cdot L_{1001}(ct)^2 ,$$

$$I^2B^2g_{22} = I \cdot I_{22} + pI \cdot L_{2332}z^2 + pI \cdot L_{2002}(ct)^2 ,$$

$$I^2B^2g_{03} = p[I \cdot (L_{0303} + L_{0033}) - I_{00} \cdot I_{33}](ct)z ,$$

$$I = \frac{3}{2}\int_0^{\pi}\int_0^1 m(c\beta)\exp[p(\gamma-1)((ct)^2 - r^2\cos^2\theta)]\beta^2\gamma^4\, d\beta \sin\theta\, d\theta$$
$$= 3\int_0^1\int_0^1 m(c\beta)\exp[p(\gamma-1)((ct)^2 - r^2u^2)]\beta^2\gamma^4 d\beta\, du\,,$$

$$I_{00} = 3\int_0^1\int_0^1 \gamma\cdot m(c\beta)\exp[p(\gamma-1)((ct)^2 - r^2u^2)]\beta^2\gamma^4\, d\beta\, du\,,$$

$$I_{33} = 3\int_0^1\int_0^1 -[1+(\gamma-1)u^2]\cdot m(c\beta)\exp[p(\gamma-1)((ct)^2 - r^2u^2)]\beta^2\gamma^4 d\beta\, du\,,$$

$$I_{11} = I_{22}$$
$$= 3\int_0^1\int_0^1 -\left[1+\frac{1}{2}(\gamma-1)(1-u^2)\right]\cdot m(c\beta)\exp[p(\gamma-1)((ct)^2 - r^2u^2)]$$
$$\times\, \beta^2\gamma^4\, d\beta\, du = -I - \tfrac{1}{2}(I_{00}+I_{33})\,,$$

$$L_{0330} = L_{3003} = -L_{0303}$$
$$= 3\int_0^1\int_0^1 -\gamma^2\beta^2u^2\cdot m(c\beta)\exp[p(\gamma-1)((ct)^2 - r^2u^2)]\beta^2\gamma^4 d\beta\, du\,,$$

$$L_{0000} = 3\int_0^1\int_0^1 \gamma^2\cdot m(c\beta)\exp[p(\gamma-1)((ct)^2 - r^2u^2)]\beta^2\gamma^4\, d\beta\, du\,,$$

$$L_{3333} = 3\int_0^1\int_0^1 [1+(\gamma-1)u^2]^2\cdot m(c\beta)\exp[p(\gamma-1)((ct)^2 - r^2u^2)]\beta^2\gamma^4\, d\beta\, du\,,$$

$$L_{1331} = L_{2332}$$
$$= 3\int_0^1\int_0^1 \frac{1}{2}(\gamma-1)^2u^2(1-u^2)\cdot m(c\beta)\exp[p(\gamma-1)((ct)^2 - r^2u^2)]$$
$$\times\, \beta^2\gamma^4 d\beta\, du = \frac{1}{2}(I - L_{3333} - L_{0330})\,,$$

$$L_{1001} = L_{2002}$$
$$= 3\int_0^1\int_0^1 -\frac{1}{2}\gamma^2(1-u^2)\beta^2\cdot m(c\beta)\exp[p(\gamma-1)((ct)^2 - r^2u^2)]$$
$$\times\, \beta^2\gamma^4\, d\beta\, du = \frac{1}{2}(I - L_{0000} - L_{0330})\,,$$

$$L_{0033} = 3\int_0^1\int_0^1 -\gamma[1+(\gamma-1)u^2]\cdot m(c\beta)$$
$$\times \exp[p(\gamma-1)((ct)^2 - r^2u^2)]\beta^2\gamma^4\, d\beta\, du$$
$$= -I - I_{33} - I_{00} + L_{0330}\,.$$

Six fundamental integrals are therefore to be calculated: I, I_{00}, I_{33}, L_{0000}, L_{3333}, L_{0330}.

4.5. CURVATURE TENSOR

Given an isotropic Riemannian G-weighted metric $ds^2 = g_{ij}dx^i dx^j$, it is natural to seek for the expression of the curvature tensor

$$K_l{}^j{}_{hk} = \frac{\partial \Gamma_l{}^j{}_h}{\partial x^k} - \frac{\partial \Gamma^j_{lk}}{\partial x^h} + \Gamma_m{}^j{}_k\Gamma_l{}^m{}_h - \Gamma_m{}^j{}_h\Gamma_l{}^m{}_k\,,$$

where $\Gamma_h{}^j{}_k$ denotes the natural affine connection, i.e. the Christoffel symbols of the second kind: $\Gamma_h{}^j{}_k = g^{lj}\Gamma_{hlk}$

Calculation of the Christoffel symbols of the first kind

$$\Gamma_{hlk} = \frac{1}{2}\left\{\frac{\partial g_{kl}}{\partial x^h} + \frac{\partial g_{lh}}{\partial x^k} - \frac{\partial g_{hk}}{\partial x^l}\right\}$$

passes through the calculation of the first partial derivatives of the g_{hk}:

$$\frac{\partial g_{ij}}{\partial x^m} = \frac{\partial}{\partial x^m}\left[\frac{1}{I^2\cdot B^2}\{I\cdot I_{ij} + p(I\cdot L_{iklj} - I_{ik}\cdot I_{lj})x^k x^l\}\right]$$
$$= \frac{1}{B^2}\left[\frac{\partial(I_{ij}/I)}{\partial x^m} + p\left[\frac{\partial(L_{iklj}/I)}{\partial x^m} - I_{ik}\cdot\frac{\partial(I_{lj}/I)}{\partial x^m} - I_{lj}\cdot\frac{\partial(I_{ik}/I)}{\partial x^m}\right]x^k x^l\right.$$
$$\left. + p\left(\frac{L_{ikmj}}{I} + \frac{L_{imkj}}{I} - \frac{I_{ik}}{I}\frac{I_{mj}}{I} - \frac{I_{im}}{I}\frac{I_{kj}}{I}\right)x^k\right].$$

For isotropic Riemannian G-weighted metrics in the local coordinate system (see the last section), complete calculations have not been achieved yet. Basically, they call for the calculations of second derivatives with respect to z and $x^0 = ct$ of the six fundamental integrals I, I_{00}, I_{33}, L_{0000}, L_{3333} and L_{0330}.

Some important relationships can be henceforth outlined:

$$\frac{\partial I}{\partial z} = 2p[I_{33} + I]z, \qquad \frac{\partial I}{\partial(ct)} = 2p[I_{00} - I](ct),$$
$$\frac{\partial I_{00}}{\partial z} = 2p[L_{0033} + I_{00}]z, \quad \frac{\partial I_{00}}{\partial(ct)} = -2p[L_{0000} + I_{00}](ct),$$
$$\frac{\partial I_{33}}{\partial z} = 2p[L_{3333} - I_{33}]z, \quad \frac{\partial I_{33}}{\partial(ct)} = 2p[L_{0033} - I_{33}](ct)\,.$$

By contrast, the derivatives of L_{0000}, L_{3333} and L_{0330} are not simply expressed as linear combinations of the six fundamental integrals.

Finally, the contravariant reciprocal coefficients g^{ij} of the g_{ij}'s are needed as well. All these calculations deserve to be undertaken in more detail.

4.6. ISOTROPIC ds^2 ON A TWO-DIMENSIONAL MINKOWSKIAN SUBSTRATE

In order to simplify the calculation of the curvature tensor the study of the two-dimensional Minkowskian substrate is tackled. The spatial motions are located along a single axis (x). The Lorentz group is a 1-parameter group: two directions are possible and the parameter $\beta = v/c$ is now supposed to vary from -1 to $+1$.

$$g\mathbf{u} = \begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \cdot \mathbf{u}, \quad \text{where} \quad \mathbf{u} = \begin{pmatrix} x \\ ct \end{pmatrix}.$$

The covariant matrix of the operation g is

$$(a_{ij}(g)) = \begin{pmatrix} -\gamma & \beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} -\operatorname{ch} w & \operatorname{sh} w \\ -\operatorname{sh} w & \operatorname{ch} w \end{pmatrix},$$

where w is the "angle of the hyperbolic rotation" g: $\beta = \operatorname{th} w$, $\gamma = \operatorname{ch} w$, $\beta\gamma = \operatorname{sh} w$. This angle varies from $-\infty$ to $+\infty$.

Since $dl_v^2 = \gamma^4 dv^2$, the measure of the Lorentz group is $dg = \gamma^2 d\beta = dw$. On the other hand,

$$\|g\mathbf{u} - \mathbf{u}\|^2 = 2(1-\gamma)[(ct)^2 - x^2] = 2(1-\gamma)\|\mathbf{u}\|^2 .$$

Under the isotropic hypothesis $\mu_{\mathbf{u},\mathbf{u}}(g) = m(c|\beta|)$, the Riemannian condition is always fulfilled, since then $d\mathrm{U}^1 = A_1 dz$, $d\mathrm{U}^0 = A_0 c dt$, where

$$A_0 = -A_1 = -\int_{-1}^{+1} m^2(c|\beta|)\gamma^3 \, d\beta = 2\int_0^{+\infty} m^2(c|\operatorname{th} w|) \operatorname{ch} w \, dw .$$

Hence

$$B^2(\mathbf{u}, d\mathbf{u}) = \frac{A_0^2}{\left[2\displaystyle\int_0^{+\infty} m(c|\operatorname{th} w|) \, dw\right]^4} .$$

The other integrals are given by

$$I = 2\int_0^{+\infty} m(c|\operatorname{th} w|) e^{p(\operatorname{ch} w - 1)\|\mathbf{u}\|^2} \, dw ,$$

$$I_{00} = -I_{11} = 2\int_0^{+\infty} m(c|\operatorname{th} w|) e^{p(\operatorname{ch} w - 1)\|\mathbf{u}\|^2} \, ch\, w \, dw ,$$

$$I_{10} = -I_{01} = 0\,,$$

$$L_{0000} = L_{1111} = -L_{0011} = -L_{1100} = 2\int_0^{+\infty} m(c|\mathrm{th}\,s|)e^{p(\mathrm{ch}\,w-1)\|u\|^2}\mathrm{ch}^2 w\,dw$$
$$= L_{0000}\,,$$

$$L_{1001} = L_{0110} = -L_{1010} = -L_{0101} = -2\int_0^{+\infty} m(c|\mathrm{th}\,w|)e^{p(\mathrm{ch}\,w-1)\|\mathbf{u}\|^2}\mathrm{sh}^2 w\,dw$$
$$= I - L_{0000}\,,$$

$$L_{0010} = L_{1000} = L_{1101} = L_{0111} = -L_{0001} = -L_{0100} = -L_{1011} = -L_{1110} = 0\,.$$

Therefore two functions are needed for the calculation of ds^2, namely, I_{00}/I and L_{0000}/I. The local coordinate system used in section 4.4. corresponds to the reference coordinate system of the two-dimensional substrate. In the formula given for ds^2 in this local coordinate system, we put $dx^1 = dx^2 = 0$, and we change dx^3 $(= dz)$ for dx. With the aid of the above relationship, one gets

$$ds^2 = \frac{1}{B^2I^2}\{g_{00}c^2dt^2 + g_{11}dx^2 + 2g_{01}cdtdx\}\,,$$

$$I^2B^2g_{00} = I\cdot I_{00} + pI\cdot(I - L_{0000})x^2 + p(I\cdot L_{0000} - I_{00}^2)(ct)^2\,,$$

$$I^2B^2g_{11} = -I\cdot I_{00} + p(I\cdot L_{0000} - I_{00}^2)x^2 + pI\cdot(I - L_{0000})(ct)^2\,,$$

$$I^2B^2g_{01} = p[-I^2 + I_{00}^2](ct)x\,.$$

Finally, the derivatives of I and I_{00} (serving to calculate the curvature tensor) are much simplified:

$$\frac{\partial I}{\partial x} = -2pI_{00}x,\quad \frac{\partial I}{\partial(ct)} = 2pI_{00}ct,\quad \frac{\partial I_{00}}{\partial x} = -2pL_{0000}x\,,\quad \frac{\partial I_{00}}{\partial(ct)} = 2pL_{0000}ct\,.$$

Remark

It is to be remarked that for two infinitely close events on the light cone ($dx = cdt$ and $x = ct$), the ds^2 remains equal to zero, like in the starting Minkowskian substrate.

4.7. NON-RIEMANNIAN ISOTROPIC G-WEIGHTED METRICS

With very general maps $m(v)$, $B^2(\mathbf{u}, \mathbf{u} + d\mathbf{u})$ depends on both $\mathbf{u}$ and $d\mathbf{u}$:

$$B^2(\mathbf{u}, d\mathbf{u}) = \frac{1}{\left[\int_G m(v)\,dg\right]^4}\cdot\frac{A_0^2 - A_\alpha^2\rho^2(d\mathbf{u})}{1 - \rho^2(d\mathbf{u})}\,,\quad \text{where}\quad \rho^2(d\mathbf{u}) = \frac{dx^2 + dy^2 + dz^2}{c^2dt^2}\,.$$

ρ can be regarded as a "3-velocity" measured in the fixed rectangular coordinate system where $\mathbf{u}$ is a 4-vector.

The definition reads

$$ds^2 = \frac{\left[\int_G m(v)\,dg\right]^4 [1-\rho^2(d\mathbf{u})]}{I^2[A_\alpha^2 - A_0^2\rho^2(d\mathbf{u})]}\{I\cdot I_{ij} + p(I\cdot L_{iklj} - I_{ik}\cdot I_{lj})x^k x^l\}dx^i dx^j\,.$$

ds^2 is no longer a quadratic form of the coordinates dx^i. Instead, ds^2 is now a ratio of two quadratic forms of the dx^i's. Like in a Finsler space, the coefficient g_{ij} of $dx^i dx^j$ depends not only on $\mathbf{u} = \{x^k\}$ but also on $d\mathbf{u} = \{dx^k\}$ [8]. This state of affairs has been already pointed out in the case of some linear representations of finite groups [5].

The dependence on $\rho(d\mathbf{u})$ shows that ds^2 is a quadratic form only in the 3D-subspace where the "3-velocity" of the displacement in $d\mathbf{u}$ is specified. In addition, the coefficients of the crossed terms $cdtdx^\alpha$ ($x^\alpha = x, y$ or z) must vanish: indeed, since $c^2dt^2 = (dx^2+dy^2+dz^2)/\rho^2$, these terms are written as

$$cdtdx^\alpha = \frac{\sqrt{dx^2+dy^2+dz^2}}{\rho}dx^\alpha$$

and do not have the form $dx^\alpha dx^\beta$. The latter condition boils down to requiring that the coordinate system (x, y, z, ct) is "synchronous" [9].

5. Case of the whole Lorentz group (including spatial rotations)

Instead of being restricted to Lorentz transformations corresponding to coordinate transformations between moving referentials (characterized by v, θ, ϕ), we might consider the whole Lorentz group including spatial rotations (characterized by the Euler's angles α', β', γ'). The hypothesis "*$m(g) \neq 0 \Rightarrow g$ is a pure Lorentz transformation (without a spatial rotation)*" is thus abandonned. Nevertheless, the isotropic condition $m(g \approx (v, \theta, \phi, \alpha', \beta', \gamma')) = m(v)$ is still assumed.

The matrices (b_{ij}) of the elements of this group are products:

$$\begin{pmatrix} 1+(\gamma-1)\sin^2\theta\cos^2\phi & (\gamma-1)\sin^2\theta\sin\phi\cos\phi & (\gamma-1)\sin\theta\cos\theta\cos\phi & -\gamma\sin\theta\cos\phi\beta \\ (\gamma-1)\sin^2\theta\sin\phi\cos\phi & 1+(\gamma-1)\sin^2\theta\sin^2\phi & (\gamma-1)\sin\theta\cos\theta\sin\phi & -\gamma\sin\theta\sin\phi\beta \\ (\gamma-1)\sin\theta\cos\theta\cos\phi & (\gamma-1)\sin\theta\cos\theta\sin\phi & 1+(\gamma-1)\cos^2\theta & -\gamma\cos\theta\beta \\ -\gamma\sin\theta\cos\phi\beta & -\gamma\sin\theta\sin\phi\beta & -\gamma\cos\theta\beta & \gamma \end{pmatrix}$$

$$\cdot\begin{pmatrix} \cos\alpha'\cos\beta'\cos\gamma' - \sin\alpha'\sin\gamma' & \sin\alpha'\cos\beta'\cos\gamma' + \cos\alpha'\sin\gamma' & -\cos\gamma'\sin\beta' & 0 \\ -\sin\alpha'\cos\gamma' - \cos\alpha'\cos\beta'\sin\gamma' & \cos\alpha'\cos\gamma' - \sin\alpha'\cos\beta'\sin\gamma' & \sin\gamma'\sin\beta' & 0 \\ \sin\beta'\cos\alpha' & \sin\alpha'\sin\beta' & \cos\beta' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $0 \leqslant \alpha' \leqslant 2\pi$, $0 \leqslant \beta' \leqslant \pi$, $0 \leqslant \gamma' \leqslant 2\pi$, it is easily verified that the integration over this group leads to

$$\int_G m^2(v) b_{ij}(g)\, dg = 0 \quad \text{(where } dg = \frac{3}{4\pi}\frac{1}{8\pi^2}\gamma^4\beta^2 \sin\theta \sin\beta'\, d\beta d\theta d\phi d\alpha' d\beta' d\gamma'\text{)}$$

except for $i = j = 0$:

$$\int_G m^2(v) b_{00}(g)\, dg = A_0 = 3\int_0^1 \gamma m^2(c\beta)\gamma^4\beta^2\, d\beta\,.$$

Thus, $\|d\mathbf{U}\|^2 = A_0^2 \cdot (cdt)^2$ cannot be proportional to $\|d\mathbf{u}\|^2$ and no Riemannian G-weighted metric can be defined from the equation applied to this group.

6. Conclusion

The mathematical formalism of the chemical algebra proves to be very general. It can be applied in a straightforward manner in the light cone of a Minkowski space endowed with the natural isometric action of the Lorentz group. The peculiar form of the ds^2 thereby defined is not physically founded, and it cannot be overemphasized that the general principles have no heuristic value *a priori*. As a complement, the calculation of curvature tensors has to be achieved. Furthermore, the study of nonisotropic ds^2 with weighting maps $\mu_{\mathbf{u},\mathbf{u}}(g) = \mu(g\mathbf{u})$ should be rapidly undertaken and compared with the preceding results concerning the occurrence of a wave function in the definition of ds^2 from an Euclidean space endowed with its group of translations [1]. Then, one may think of possible physical speculations for the parameter p ("curvature variable" of dimension $(\text{length})^{-2}$?) and for the definition equation of ds^2 (a space-time model would be constructed from a flat *model* mapped by a scalar "wave function": this model would be curved by a network of connections corresponding to the potential motions gathered in a group G and weighted by a map $\mu_{\mathbf{u},\mathbf{u}}(g)$?). In particular, it must stressed that in the derivation of ds^2, the coordinate system is not merely a simple numerical reference but refers to a *model* as a rectangular trihedron-clock of the Minkowski space. However, these representations are still far too speculative to be detailed. It is just to be reminded that all these developments come from the algebraic analysis of chemical pairing equilibria

$$2\,MN \rightleftarrows MM + NN\,,$$

and more precisely from their thermodynamical constant

$$K = \frac{[MM]\cdot[NN]}{[MN]^2}\,.$$

Through a process of mathematical generalization, it has been "applied" when M

and N represent "events in a space-time" instead of molecules in a flask .... But the meaning of the "pairs MM, NN and MN" as well as of the "interconversion $\rightleftarrows$" has been lost .... The superficial approach here reported surely deserves further investigations, and these are in progress.

Reverences and notes

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[4] R. Chauvin, Paper V of this series, J. Math. Chem. 17 (1995) 235.

[5] R. Chauvin, Paper IV of this series, J. Math. Chem. 16 (1994) 285.

[6] A speculative physical justification of the assumption $m \neq 1$ is outlined. No limits for translations $\mathbf{v}t$ were enforced in the classical Euclidean space [1]: the same "weight" $m(\mathbf{v}t) = 1$ was thus assigned to all the translations $\mathbf{v}t$. In contrast, the limits $0 \leqslant v \leqslant c$ enforced in the Minkowski space suggest that the "weight" $m(g)$ of $\mathbf{v}$ might vary continuously and equal zero for $v \geqslant c$.

[7] R. Chauvin, Paper I of this series, J. Math. Chem. 16 (1994) 245.

[8] D. Lovelock and H. Rund, *Tensors, Differential Forms and Variational Principles* (Dover, New York, 1989).

[9] L. Landau and E. Lifchitz, Théorie des Champs, *Physique Théorique*, Vol. 2, 4th French Ed. (Mir, Moscow, 1989).